ANOMALOUS LIGHT ABSORPTION BY A TURBULENT PLASMA

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When a plasma is penetrated by an electromagnetic wave at a frequency close to the plasma electron frequency, Langmuir and ion-sound waves can be excited within the plasma. Under these conditions the linear light absorption mechanism becomes ineffective and the plasma itself transforms to a turbulent state [1]. Such conditions can be created, for example, in a light detonation plasma.

At radiant energy flux densities $I \gtrsim 10^8$ W/cm² a light detonation charge driven by a shock wave can be maintained in air [2]. A light detonation wave (LDW) contains a shock wave (density discontinuity) and a layer of thickness ℓ, in which the light energy flux is absorbed. Usually $\ell << d$ (where d is the characteristic transverse dimension of the discharge front, which is of the order of magnitude of the size of the light beam). The plane discharge front moves toward the light ray; the speed of the LDW is determined by conditions of conservation of mass flux density, momentum, and energy on the density discontinuity. The theoretically calculated value of the LDW velocity agrees well with experimental data:

$$D = \left[2\left(\gamma_0^2 - 1\right) I/\rho_0\right]^{1/3} \tag{0.1}$$

(γ_0 is the adiabatic index and ρ_0 is the density of the cold gas [2]). As a rule, the velocity of discharge front motion $D \gtrsim 10$ cm/sec, the plasma temperature $T_e \gtrsim 10$ eV, and the electron density in the plasma reaches values at which the electromagnetic wave frequency becomes equal to the plasma electron frequency.

In a turbulent plasma for a sufficiently high level of the fluctuation fields collective absorption of light energy can reach maximal values. We will demonstrate that this mechanism can play the defining role in a light detonation discharge.

1. Plasma Wave Dynamics. We will assume that a transverse electromagnetic wave linearly polarized along the x-axis

$$E_t = E_0 \exp(ik_t z - i\omega_t t) + \text{c.c.} \tag{1.1}$$

propagates along the z-axis toward the discharge. For slight detuning ($|\omega_t - \Omega_e| \ll \Omega_e$), such that the frequency ω_t is close to the electron plasma frequency Ω_e and the wave number $k_t \to 0$, parametric effects leading to development of intense Langmuir and ion-sound waves manifest themselves most intensely [3]. We will now consider a reference frame attached to the discharge front. The equations for the plasma waves can be obtained from the hydrodynamic equations of the plasma and the Poisson equation:

$$\begin{gathered}\frac{\partial}{\partial t}\delta n_j + n_0\frac{\partial}{\partial x}v_j + \frac{\partial}{\partial x}(\delta n_j v_j) = 0,\\ \frac{\partial}{\partial t}v_j + v_j\frac{\partial}{\partial x}v_j = \frac{eE}{m_j} - \frac{\gamma T_j}{m_j(n_0 + \delta n_j)}\frac{\partial}{\partial x}\delta n_j,\\ \frac{\partial E}{\partial x} = -4\pi e(\delta n_e - \delta n_i).\end{gathered} \tag{1.2}$$

Here v_j, δn_j ($j = e$, i) are plasma velocity and density fluctuations for electrons and ions, n_0 is the mean density, E is the electric field. For Langmuir waves L the adiabatic $\gamma = 3$, while for ion-sound waves s$\gamma = 1$. We assume that the electron temperature T_e is much greater than the ion temperature T_i, and that in the equation of motion for the ions the term related to the pressure gradient can be neglected.

Electrons in the plasma field E_p can participate in L- and s-oscillations, while ions can carry out only s-oscillations, i.e.,

$$\begin{gathered}\delta n_e = \delta n_s + \delta n_L,\ v_e = v_s + v_L,\ \delta n_i = \delta N_s,\\ v_i = V_s,\ E_p = E_s + E_L.\end{gathered} \tag{1.3}$$

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Substituting Eq. (1.3) in Eq. (1.2) we find a system of equations describing the plasma wave dynamics:

$$\left(\frac{\partial^2}{\partial t^2}+\Omega_e^2-3V_T^2\frac{\partial^2}{\partial x^2}\right)\delta n_L=\frac{n_0}{2}\frac{\partial^2}{\partial x^2}v_e^2-\frac{3}{2}\frac{V_T^2}{n_0}\frac{\partial^2}{\partial x^2}\delta n_e^2-\frac{\partial^2}{\partial x\,\partial t}(\delta n_e v_e),$$
$$\left(\frac{\partial^2}{\partial t^2}-c_s^2\frac{\partial^2}{\partial x^2}\right)\delta n_s=\frac{n_0}{2}\frac{\partial^2}{\partial x^2}\left(v_i^2+\frac{m_e}{m_i}v_e^2-\frac{c_s^2}{n_0^2}\delta n_e^2\right)-$$
$$-\frac{\partial^2}{\partial x\,\partial t}\left(\delta n_i v_i+\frac{m_e}{m_i}\delta n_e v_e\right) \tag{1.4}$$

($V_T=\sqrt{T_e/m_e}$ is the electron thermal velocity, $c_s=\sqrt{T_e/m_i}$ is the speed of the ion-sound wave).

Considering that the electromagnetic wave sets the electrons into motion along the x-axis:

$$v_t=\frac{-ie}{\omega_t m_e}E_0\exp(-i\omega_t t)+\text{c.c.}, \tag{1.5}$$

we will seek a solution of Eq. (1.4) with consideration of nonlinear interaction of the three waves which generates higher harmonics and modulation modes. Thereform in the future we will represent all the variable quantities in the form of series:

$$v_i=\varepsilon V_s^{(1)}e^{i\Phi_s}+\varepsilon^2V_s^{(2)}e^{2i\Phi_s}+\ldots+\text{c.c.},$$
$$v_e=\varepsilon\left(v_L^{(1)}e^{i\Phi_L}+v_s^{(1)}e^{i\Phi_s}\right)+\varepsilon^2\left(v_t+v_L^{(2)}e^{2i\Phi_L}+v_s^{(2)}e^{2i\Phi_s}+\right.$$
$$\left.+v_+e^{i(\Phi_L+\Phi_s)}+v_-e^{i(\Phi_L-\Phi_s)}\right)+\ldots+\text{c.c.}, \tag{1.6}$$

where $\Phi_L=k_Lx-\omega_Lt$; $\Phi_s=k_sx-\omega_st$; ω_L, ω_s, k_L, k_s are the frequencies and wave numbers of L- and s-waves; v_t is defined by Eq. (1.5).

Substituting the solutions of Eq. (1.6) in Eq. (1.4) and considering the exact resonance condition (for $k_t\approx0$) $|k_L|=-|k_s|$, $\omega_t=\omega_L+\omega_s$, after averaging over high frequency oscillations we arrive at a system of abbreviated equations

$$\left(\frac{\partial}{\partial t}-V_L\frac{\partial}{\partial x}\right)a_L-i\alpha_L|a_L|^2a_L-i\beta_L|a_s|^2a_L=-\lambda_La_s^*,$$
$$\left(\frac{\partial}{\partial t}+c_s\frac{\partial}{\partial x}\right)a_s-i\alpha_s|a_s|^2a_s-i\beta_s|a_L|^2a_s=-\lambda_sa_L^*. \tag{1.7}$$

Here $V_L=3(kV_T/\Omega_e)V_T$ is the group velocity of the Langmuir wave; a_L, a_s are complex amplitudes of δn_L, δn_s; α_j, β_j, λ_j (j = L, s) are nonlinearity coefficients and coupling parameters:

$$\alpha_L=\frac{1}{4}\frac{\Omega_e}{n_0^2},\quad \beta_L=\frac{\Omega_e}{8n_0^2}\frac{\Omega_e^2}{3k^2V_T^2+2\Omega_e\omega_s},$$
$$\alpha_s=\frac{\omega_s}{8n_0^2}\frac{\Omega_e^2}{k^2V_T^2},\quad \beta_s=\frac{\Omega_e}{8n_0^2}\frac{m_e\Omega_e}{m_i\omega_s}\frac{\Omega_e^2}{3k^2V_T^2+2\Omega_e\omega_s},$$
$$\lambda_L=\frac{1}{2}kv_t^0,\quad \lambda_s=\frac{1}{2}\frac{m_e\Omega_e}{m_i\omega_s}kv_t^0,\quad v_t^0=\frac{eE_0}{m_e\omega_t}. \tag{1.8}$$

We will note that for α_j, β_j = 0 Eq. (1.7) describes the initial stage of parametric instability with increment $\sqrt{\lambda_L\lambda_s}=kv_t^0\sqrt{\frac{m_e\Omega_e}{m_i\omega_i}}=(eE_0/m_e\omega_tV_T)(\Omega_e\omega_s)^{1/2}$ [4]. The increment is proportional to the amplitude of the pump wave. In the literature such instability is also called decay instability.

We will seek a solution of Eq. (1.7) in the form of a wave with constant profile. We rewrite Eq. (1.7) in the new variable ξ = x − Vt, obtaining for standing waves

$$\frac{\partial a_L}{\partial\xi}-i\alpha_L|a_L|^2a_L-i\beta_L|a_s|^2a_L=-\lambda_La_s^*,$$
$$\frac{\partial a_s}{\partial\xi}-i\alpha_s|a_s|^2a_s-i\beta_s|a_L|^2a_s=-\lambda_sa_L^*, \tag{1.9}$$

where the coefficients are normalized in the following manner:

$$\alpha_j \to \alpha_j/u_j,\ \beta_j \to \beta_j/u_j,\ \lambda_j \to \lambda_j/u_j,\ u_L = -(V_L + V),$$
$$u_s = c_s - V,\ j = L, s.$$

In Eq. (1.9) we transform to amplitude-phase variables, taking $a_j = A_j \exp(i\varphi_j)$. Then Eq. (1.9) appears as

$$\frac{d}{d\xi} A_L^2 = -2\lambda_L A_L A_s \cos\varphi,\quad \varphi = \varphi_L + \varphi_s,\quad \frac{d}{d\xi} A_s^2 = -2\lambda_s A_L A_s \cos\varphi,$$
$$\frac{d}{d\xi}\varphi = \alpha_L A_L^2\left(1 + \frac{\alpha_s A_s^2}{\alpha_L A_L^2}\right) + \beta_s A_L^2\left(1 + \frac{\beta_L A_s^2}{\beta_s A_L^2}\right) + \lambda_L \frac{A_s}{A_L}\left(1 + \frac{\lambda_s A_L^2}{\lambda_L A_s^2}\right)\sin\varphi. \tag{1.10}$$

From the first two equations of (1.10) we find the integral of the motion

$$\lambda_L A_s^2 - \lambda_s A_L^2 = C. \tag{1.11}$$

Below we will regard the integration constant C = 0, assuming that as $\xi \to \pm\infty$ $A^2{}_j \to 0$.

Making use of Eq. (1.11), from Eq. (1.10) we obtain the equations of wave dynamics:

$$\frac{d^2}{d\xi^2}\varphi - 2\lambda_L\lambda_s \sin 2\varphi = 0,$$
$$\alpha A_L^2 = \frac{d\varphi}{d\xi} - 2\sqrt{\lambda_L\lambda_s}\sin\varphi,\quad \alpha = \alpha_L\left(1 + \frac{\alpha_s\lambda_s}{\alpha_L\lambda_L}\right) + \beta_s\left(1 + \frac{\beta_L\lambda_s}{\beta_s\lambda_L}\right).$$

The first of these has the form of a steady state sine–Gordon equation, solutions of which are well known and expressible in terms of elliptical functions [5]. We write the solution on the separatrix for boundary conditions $\xi \to \pm\infty$, $A^2{}_j \to 0$, corresponding to the solution

$$A_L = \left(4\frac{\sqrt{|\lambda_L\lambda_s|}}{|\alpha|}\right)^{1/2} \mathrm{ch}^{-1/2}\left(2\sqrt{|\lambda_L\lambda_s|}(x - Vt)\right).$$

2. System Stochastization. Spectral Properties. We will study the system dynamics over time close to the separatrix under the action of a periodic disturbance. To do this we consider the nonresonant terms in Eq. (1.4) with a higher order of smallness: $\partial^2\delta n_e v_e/\partial x\partial t = 2kv_t{}^0\Omega_e a_L$, $(\partial^2\delta n_e v_e/\partial x\partial t)(m_e/m_i) = (m_e/m_i)\, k v_t^0 a_s$. After corresponding averaging we find the system of abbreviated equations

$$\dot a_L - i\alpha_L|a_L|^2 a_L - i\beta_L|a_s|^2 a_L = -\lambda_L a_s^* + 2\varepsilon\lambda_L a_L \exp(-i\Omega_e t),$$
$$\dot a_s - i\alpha_s|a_s|^2 a_s - i\beta_s|a_L|^2 a_s = -\lambda_s a_L^* + \varepsilon\lambda_s a_s \exp(-i\Omega_e t), \tag{2.1}$$

where α_j, β_j, λ_j are defined by Eq. (1.8); $\dot a_j \equiv \partial a_j/\partial t$.

In the variables $a_j = A_j \exp(i\varphi_j)$ system (2.1) can be rewritten in the form

$$\ddot\psi + \omega_0^2 \sin\psi = \varepsilon\omega_0^2 \cos\Omega_e t,\quad \varepsilon = \omega_0/\Omega_e; \tag{2.2}$$
$$A_L^2 = \frac{1}{2\alpha}\left(\dot\psi + 2\omega_0 \sin\frac{\psi}{2}\right); \tag{2.3}$$
$$\omega_0^2 = 4\lambda_L\lambda_s,\quad \psi = 2\varphi + \pi. \tag{2.4}$$

In deriving Eq. (2.2) use was made of Eq. (1.11), with averaging over high frequency oscillations carried out with the assumption that $\omega_0/\Omega_e \ll 1$.

We will consider ψ as a generalized motion coordinate. Then the unperturbed portion of Eq. (2.2) has the form of the equation of a nonlinear pendulum of unit mass with Hamiltonian

$$H = \frac{1}{2}\dot\psi^2 - \omega_0^2\cos\psi. \tag{2.5}$$

The solution on the separatrix corresponding to the Hamiltonian $H_C = \omega_0{}^2$ and the boundary condition $A_L{}^2 \to 0$ as $t \to \infty$, is

$$A_L^2 = \frac{2\omega_0^2}{\alpha} \operatorname{ch}^{-1}\omega_0 (t - t_0), \quad \dot{\psi} = 2\omega_0 \operatorname{ch}^{-1}\omega_0 (t - t_0) \tag{2.6}$$

(t_0 is an integration constant). We introduce the parameter

$$N = \omega_0/\omega(H) \tag{2.7}$$

(ω_0 is the eigenfrequency of small oscillations).

It was shown in [6] that near the separatrix

$$N \sim \frac{1}{\pi} \ln \frac{32H_c}{H_c - H}, \quad H_c = \omega_0^2, \tag{2.8}$$

and the Fourier harmonic amplitude for $\dot{\psi}$ is

$$b_n \sim 8\omega \exp(-\pi n/N),$$

i.e., all amplitudes are approximately equal up to $n \sim N$. With approach to the separatrix $N \to \infty$, and the spectrum tends to continuity.

We will obtain an expression for the spectral power density of the amplitude A_j. We now introduce the correlation function $q_j(\tau) = \langle A_j(t)A_j(t + \tau)\rangle$ (the angle brackets denote averaging over the set (time) [6]). Then the spectral density

$$q_j(\omega) = \int_{-\infty}^{+\infty} d\tau \exp(i\omega\tau)\, q_j(\tau).$$

Using the expression for A_L from Eq. (2.6) we find

$$q_L(\tau) = \frac{2\pi}{\alpha} \operatorname{ch}^{-1}\omega_0\tau, \quad q_L(\omega) = \frac{\pi^2}{\alpha\omega_0} \operatorname{ch}^{-1} \frac{\pi\omega}{2\omega_0}.$$

Hence it is evident that the wave spectrum is wide (the spectral width is approximately equal to the eigenfrequency $\Delta\omega = \omega_0$) and decays exponentially for high ω.

To study the disturbed system (2.2)-(2.4) we transform to the variables action J – angle θ with Hamiltonian

$$H = H_0(J) + \varepsilon V(J, \theta) \cos \Omega_e t, \quad V = \omega_0^2 \psi, \tag{2.9}$$

where $H_0(J)$ is defined by Eq. (2.5) with ψ, $\dot{\psi}$ replaced by J, θ.

In these variables Eq. (2.2) takes on the form

$$\dot{J} = -\frac{\varepsilon}{\omega(J)} \frac{\partial V}{\partial \psi} \dot{\psi} \cos \Omega_e t, \quad \dot{\theta} = \omega(J) + \varepsilon \frac{\partial V}{\partial J} \cos \Omega_e t. \tag{2.10}$$

We write the reflection along the separatrix for system (2.10) as [5]

$$\begin{gathered} \bar{J} = J + \frac{\varepsilon}{\omega(J)} C(J, \chi), \quad \bar{\chi} = \chi + \frac{\pi\Omega_e}{\omega(J)} - \frac{\pi\Omega_e}{\omega^3} \varepsilon \left|\frac{\partial\omega}{\partial J}\right| C(J, \chi), \\ C(J, \chi) = -\int_{\Delta t} dt \frac{\partial V}{\partial \psi} \dot{\psi} \cos \chi(t), \quad \dot{\chi} = \Omega_e. \end{gathered} \tag{2.11}$$

From Eq. (2.11) we define the parameter K, which characterizes the stochastic state of the system:

$$K = \frac{\varepsilon\pi\Omega_e}{\omega^3} \left|\frac{\partial\omega}{\partial J}\right| C_0, \quad C_0 = \left|\frac{\partial C}{\partial \chi}\right| = \left|\int_{\Delta t} dt \frac{\partial V}{\partial \psi} \dot{\psi} \sin \Omega_e t\right|. \tag{2.12}$$

For $\Omega_e \gg \omega_0$ near the separatrix it follows from Eqs. (2.7), (2.8) that

$$\left|\frac{\partial\omega}{\partial J}\right| = \omega\left|\frac{\partial\omega}{\partial H}\right| \sim \frac{\omega^3}{\pi\omega_0}\frac{1}{|H-H_c|}, \quad K \sim \varepsilon\frac{\Omega_e}{\omega_0}\frac{C_0}{|H-H_c|}. \tag{2.13}$$

Substituting in Eqs. (2.12), (2.13) the values of $\dot{\psi}$, V from Eqs. (2.6), (2.9), and considering that intense stochasticity develops for $K \gtrsim 1$, we find the level of stochasticity in H:

$$\frac{|H-H_c|}{H_c} \leqslant \varepsilon\frac{\Omega_e}{\omega_0}\exp\left(-\frac{\pi\Omega_e}{2\omega_0}\right).$$

The physical pattern of developing stochasticity is related to nonlinear resonances within the the system under the action of the periodic disturbance. In fact, the resonance condition has the form $m\omega(H) = \Omega_e$, and the distance between resonances $\delta\omega = |\omega(H_{m+1}) - \omega(H_m)| \simeq \omega^2/\Omega_e$. The width of the resonance in frequency is $\Delta\omega = |d\omega/dH|\Delta H$. With approach to the separatrix the frequency $\omega(H) \to 0$, the derivative thereof increases ($|d\omega/dH| \sim \exp(\pi\omega_0/\omega)$), so that the width of resonances increases, and the distance between resonances decreases. As a result the network of resonances overlaps and a phase layer with stochastic dynamics develops.

To this point we have in fact determined the region of wave phase stochasticity. We will now evaluate the stochasticity of the plasma waves with respect to energy (amplitude). It follows from Eq. (2.6) that the stochasticity region $\delta A^2/A^2 \simeq |d\omega/dH|(\Delta H/\omega_0)$. Substituting in this expression the values from Eqs. (2.7), (2.8), (2.13), we have

$$(\delta A^2/A^2) \sim \varepsilon^2 = (\omega_0^2/\Omega_e^2).$$

3. Evaluation of Results. We will first find the characteristic frequency and relative fluctuation level. Using in Eqs. (2.4), (2.6) the values of the coefficients from Eq. (1.8), we obtain

$$\omega_0 = kv_t^0\sqrt{\frac{m_e\Omega_e}{m_i\omega_s}},$$

$$\frac{|\delta n_e|_L^2 + |\delta n_e|_s^2 + |\delta n_i|^2}{n_0^2} \simeq 48\frac{kv_t^0}{\Omega_e}\left(\frac{m_i\omega_s}{m_e\Omega_e}\right)^{3/2}\frac{k^2V_T^2}{\Omega_e^2}\left(1 + 2\frac{m_e\Omega_e}{m_i\omega_s}\right). \tag{3.1}$$

It is evident that ω_0 coincides with the value of the plasma oscillation increment in linear theory (initial stage of decay).

We will apply the results obtained to a light detonation discharge. Assuming that the LDW is maintained by CO_2-laser radiation, so that $\omega_t \approx \Omega_e \approx 2\cdot10^{14}$ sec^{-1}, and taking $kV_T \sim 0.1\,\Omega_e$, $T_e \sim 10$ eV, $V_T \sim 10^8$ cm/sec, we find from Eq. (3.1) $\omega_0 \sim 10^{11}$ sec^{-1}, $\delta n/n_0 \sim (10^{-1} - 10^{-2})$, with characteristic wave numbers $k \sim 10^5$ cm^{-1}. Usually in experiment the transverse discharge diameter $d \sim 1$ cm [2, 7] and the condition $kd \gg 1$ is well satisfied. The diffraction angle for plasma waves depends on λ/ℓ and for $\ell \sim 10^{-2}$ cm $\lambda/\ell \ll 1$, i.e., it is small.

We will now consider the rate electromagnetic energy contribution to the plasma. For a linear absorption mechanism the energy contribution is determined by the frequency of collision of electrons with ions ν_s. It was shown in [4] that it is not only the micro-fields of individual particles, but the electric field of plasma oscillations which has scattering properties. Calculations show that for field values of the order of magnitude of the thermal field, the electron path length due to scattering on the thermodynamic equilibrium plasma oscillations is an order of magnitude greater than the path for paired collisions. Correspondingly the effective frequency of energy contribution caused by scattering on plasma oscillations is less than ν_s. But with instability the oscillation amplitude rises to values many times the equilibrium value. In such cases the free path length and effective energy contribution frequency depend upon scattering on plasma oscillations. Such heating is often termed turbulent, since the mechanism controlling the nature of the anomalous resistance is turbulence produced by the instability. In order of magnitude the effective frequency characterizing the rate of the turbulent contribution has the form $-\nu_T \sim \Omega_e W_T/n_0 T_e$ ($W_T \sim \bar{E}^2/8\pi + n_0 m_e \bar{v}_e^2/2$ is the energy density of the turbulent oscillations). The mean values of potential and kinetic energy for longitudinal oscillations are approximately equal and $W_T = \bar{E}^2/4\pi \sim (\delta n_e)^2 T_e/n_0$. Then $\nu_T \sim \Omega_e(\delta n_e/n_0)^2$ [4]. The studies performed show that for typical discharge parameters ($T_e \sim 10$ eV, $n_0 \sim 10^{19}$ cm^{-3} the turbulent energy channel is more effective ($\nu_s \sim n_e v_T \sigma$, $\nu_s \sim (10^9\text{-}10^{10})$ sec^{-1}, $\nu_T \sim (10^{12}\text{-}10^{13})$ sec^{-1}).

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ASYMPTOTE OF THE NAVIER–STOKES EQUATION SOLUTION IN THE VICINITY OF A BOUNDARY ANGLE

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In a study of single-sided limitations for the Navier–Stokes equations, [1] considered the function $\psi(r, \varphi)$, which satisfies the equation

$$\Delta\Delta\psi = 0,\ r < \varepsilon,\ -\pi < \varphi < 0 \tag{1}$$

(where $\varepsilon > 0$ is a constant) with boundary conditions

$$\psi = 0,\ \Delta\psi = 0,\ \varphi = 0,\ 0 < r < \varepsilon,$$
$$\psi = 0,\ \frac{\partial\psi}{\partial\varphi} = r,\ \ \varphi = -\pi,\ \ 0 < r < \varepsilon.$$

Here (r, φ) is a planar polar coordinate system and Δ is the Laplace operator. In addition we assume the function belongs to the Sobolev space W_2^2 in the semicircle $S_\varepsilon = \{(r, \varphi) : r < \varepsilon, -\pi < \varphi < 0\}$. Using the method developed in [2, 3] the authors presented the expression

$$\psi = -r\sin\varphi + Ar^{3/2}\left(\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2}\right) + O(r^2\ln r) \tag{2}$$

for $r \to 0$, $-\pi < \varphi < 0$, $A =$ const, which is dependent on ψ. Asymptotic representations of $\partial\psi/\partial r$, $\partial\psi/\partial\varphi$, $\Delta\psi$ can be obtained from Eq. (2) by formal differentiation. In fact, Eq. (2) can be refined: for ψ one can expand in an asymptotic series [2, 4]

$$\psi = -r\sin\varphi t\sum_{j=3}^{\infty} A_j r^{j/2}\Phi_j(\varphi),\quad A_j = \text{const}, \tag{3}$$

where Φ_j are eigenfunctions, normalized in $L_2[-\pi, 0]$, of the problem

$$\frac{1}{4}j^2\left(\frac{j}{2}-2\right)^2\Phi + \frac{j^2}{2}\Phi''' + \Phi^{IV} = 0,$$
$$-\pi < \varphi < 0,\ \Phi(-\pi) = \Phi(0) = 0,\ \Phi'(-\pi) = \Phi''(0) = 0. \tag{4}$$

Equation (3) is asymptotic in the sense that no matter what the value of N, the estimates

$$\left|D^\alpha(\psi) + r\sin\varphi - \sum_{j=3}^{N} A_j r^{j/2}\Phi_j(\varphi)\right| = O(r^{(N+1)/2-|\alpha|})$$

are valid as $r \to 0$ for all α. Here $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}$; $|\alpha| = \alpha_1 + \alpha_2$. Note that Eq. (4) with constant coefficients is easily solved and the eigenfunctions of Eq. (4) can be written explicity; Equation (3) is a special case of a more general expression which gives the asymptotic representation of a boundary problem for an arbitrary elliptic equation in the vicinity of an angular point on the region's boundary. It follows from Eq. (3) that in Eq. (2) the

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